THE FREE CONVECTION OF A CONDUCTING FLUID IN CONNECTED VERTICAL CHANNELS

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The convection of a conducting fluid in vertical channels in a magnetic field has been studied in a series of papers (see review [1]). Motion in isolated channels was considered in these papers. The present paper solves the problem of convection in a system of two connected plane vertical channels. A solution is obtained for the problem of steady-state motion when heating occurs from the side. Equilibrium stability is also investigated for a liquid which is heated from below (the magnetohydrodynamic generalization of the problem considered previously [2]).

1. **Steady-state motion.** Two parallel plane vertical channels of width 2h, separated by a dielectric layer of thickness 2(d − h) (see Fig. 1), are filled with a conducting fluid and placed in an external magnetic field H_0, perpendicular to the channel boundaries. The external channel boundaries are maintained at constant temperatures of $\pm\Theta$. The channels are joined both above and below, so that when convection of the fluid occurs it may rise in one channel and fall in the other (a model of the middle part of a long convection loop). The equations for convection of a conducting fluid in a magnetic field have the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla\left(p + \frac{H^2}{8\pi}\right) + $$

$$+ \frac{1}{4\pi\rho_0}(\mathbf{H}\nabla)\mathbf{H} + \nu\Delta\mathbf{v} - \mathbf{g}\beta T, \qquad (1.1)$$

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{v}\nabla)\mathbf{H} = (\mathbf{H}\nabla)\mathbf{v} + \frac{c^2}{4\pi\sigma}\Delta\mathbf{H}, \qquad (1.2)$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \chi\Delta T,$$

$$\operatorname{div}\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{H} = 0 \qquad (1.3)$$

(all the symbols are those in general use). We shall seek a steady-state solution of the system of equations (1.1)–(1.3) in the following form:

$$v_x = v_y = 0, \quad v_z = v(x), \quad T = T(x),$$
$$H_x = H_0, \ H_y = 0, \ H_z = H(x), \ p = p(x, z). \qquad (1.4)$$

We obtain the following equations for the velocity v, temperature T, induced magnetic field H, and pressure p:

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(p + \frac{H^2}{8\pi}\right) = \frac{1}{4\pi\rho_0}H_0H' + \nu v'' + g\beta T,$$

$$\frac{1}{\rho_0}\frac{\partial}{\partial x}\left(p + \frac{H^2}{8\pi}\right) = 0, \qquad (1.5)$$

$$H_0 v' + \frac{c^2}{4\pi\sigma}H'' = 0, \quad T'' = 0, \quad T_m'' = 0,$$
$$H_m'' = 0. \qquad (1.6)$$

Here T_m and H_m are the temperature and induced field in the solid nonconducting layer between the channels; the primes denote differentiation with respect to x.

Equations (1.5)–(1.6) are now written in dimensionless form, taking h, ν/h, Θ, and H_0 as units of distance, velocity, temperature, and field, respectively.

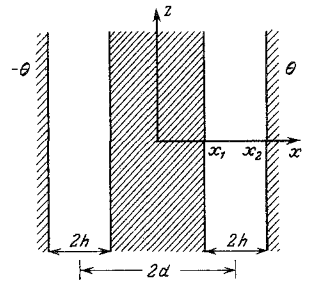

Fig. 1

The dimensionless velocity, temperature, and field are

$$v'' + GT + M^2P_m^{-1}H' = C, \qquad (1.7)$$

$$H'' + P_m v' = 0, \qquad T'' = 0, \qquad T_m'' = 0,$$
$$H_m'' = 0. \qquad (1.8)$$

Here C is the constant of separation for the variables. Three dimensionless parameters enter into the equations: the Grashof number G, the Hartmann number M, and the magnetic Prandtl number P_m, which are equal to

$$G = \frac{g\beta\Theta h^3}{\nu^2},$$

$$M = \frac{H_0h}{c}\left(\frac{\sigma}{\rho_0\nu}\right)^{1/2}, \quad P_m = \frac{4\pi\sigma\nu}{c^2}. \qquad (1.9)$$

At the channel boundaries the fluid velocity vanishes, the temperature at the outer boundaries is given, and at the boundaries between the channel and the interlayer the temperature and heat flux are continuous; the induced field disappears at the outer boundaries and is continuous at the boundaries between the fluid and the interlayer. Thus the boundary conditions are

$$v = 0, \quad T = T_m, \quad \lambda T' = T_m',$$

$$H = H_m \quad \text{for } x = x_1 = \pm\frac{d-h}{h},$$

$$v = 0, \quad T = \pm 1,$$

$$H = 0 \quad \text{for } x = x_2 = \pm\frac{d+h}{h} \qquad \left(\lambda = \frac{\varkappa}{\varkappa_m}\right). \qquad (1.10)$$

Here and in what follows the plus and minus signs refer to the right-hand and left-hand channels, respectively; $\varkappa$ and $\varkappa_m$ are the thermal conductivities of the fluid and the interlayer.

Solving linear equations (1.7)–(1.8) together with boundary conditions (1.10) and allowing for the fact that the flow occurs in a closed circuit (the rate of fluid flow over the cross section of both channels is equal to zero), we obtain the following distributions of temperature, velocity, and field:

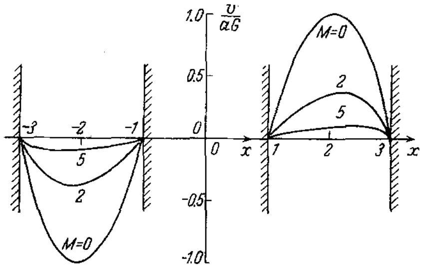

Fig. 2

$$T_m = a\lambda x, \qquad T = a\,[(\lambda - 1)\,x_1 + x],$$

$$a = (2 + \lambda\,|\,x_1|)^{-1}, \tag{1.11}$$

$$v = \frac{aG}{M^2}\left[x - \frac{x_1\,\mathrm{sh}\,M\,(x_2 - x)}{\mathrm{sh}\,M\,(x_2 - x_1)} - \frac{x_2\,\mathrm{sh}\,M\,(x - x_1)}{\mathrm{sh}\,M\,(x_2 - x_1)}\right], \tag{1.12}$$

$$H = \frac{aGP_m}{M^2}\left\{\frac{x_2^2 - x^2}{2} + \frac{x_1[1 - \mathrm{ch}\,M\,(x_2 - x)]}{M\,\mathrm{sh}\,M\,(x_2 - x_1)} + \right.$$

$$\left. + \frac{x_2\,[\mathrm{ch}\,M\,(x - x_1) - \mathrm{ch}\,M\,(x_2 - x_1)]}{M\,\mathrm{sh}\,M\,(x_2 - x_1)}\right\}, \tag{1.13}$$

$$H_m = \frac{aGP_m}{M^2}\left\{\frac{x_2^2 - x_1^2}{2} + \right.$$

$$\left. + \frac{(x_1 + x_2)\,[1 - \mathrm{ch}\,M\,(x_2 - x_1)]}{M\,\mathrm{sh}\,M\,(x_2 - x_1)}\right\}. \tag{1.14}$$

It is clear from (1.11) that the temperature distribution is independent of the field.

Velocity profiles for the case in which the thickness of the interlayer is equal to the channel width (i.e., $|x_1| = 1$) are given in Fig. 2 for certain values of the Hartmann number M. The velocity in the absence of a field is obtained from (1.12) at the limit as $M \to 0$:

$$v = \frac{aG}{6\,(x_2 - x_1)}\,[x\,(x_2 - x_1)^3 -$$

$$- x_1\,(x_2 - x)^3 - x_2\,(x - x_1)^3]\,. \tag{1.15}$$

We see that the rate of motion is determined by the parameter $aG = G(2 + \lambda|x_1|)^{-1}$. As the field increases the motion slows down and Hartmann boundary layers are formed in the flow at the walls. We note, however, that the velocity profile in each channel differs in form from the well-known profile of the Hartmann case, which is explained by the nonuniformity of the mass (convective) force over the channel cross section.

The induced field is an even function of the transverse coordinate x. The induced-current density in the fluid $j = (c/4\pi)$ rot H can be distributed over the channel cross section in the same way as the velocity.

In the flow being considered, convective heat transfer occurs upward along each channel. The total heat flux (per unit length along the y-axis) is determined from the formula

$$Q = \rho c_p \int vT\,dx\,. \tag{1.16}$$

Here v and T are the dimensional velocity and temperature, c_p is the heat capacity of the fluid, and the integration is carried out over the cross sections of both channels.

After substituting v and T into (1.16) we have

$$Q = \frac{4\rho c_p g\beta\Theta^2 h^3}{(2 + \lambda x_1)^2\,\nu M^2}\Big[x_1\,(x_1 + 1)\,(\lambda - 1)\Big(1 - \frac{\mathrm{th}\,M}{M}\Big) +$$

$$+ \frac{1}{3}\,(3x_1^2 + 6x_1 + 4) -$$

$$- x_1\,(x_1 + 2)\,\frac{\mathrm{th}\,M}{M} - \frac{2\,\mathrm{cth}\,2M}{M} + \frac{1}{M^2}\Big]. \tag{1.17}$$

The thermal flux decreases as the field increases. For weak fields ($M \ll 1$) from (1.17) we have

$$Q = Q_0\left[1 - \frac{2}{21}\,\frac{64 + 63x_1 + 63x_1\,(x_1 + 1)\,\lambda}{16 + 15x_1 + 15x_1\,(x_1 + 1)\,\lambda}\,M^2 + \ldots\right],$$

$$Q_0 = \frac{4}{45}\,\frac{\rho c_p g\beta\Theta^2 h^3}{\nu}\,\frac{16 + 15x_1 + 15x_1\,(x_1 + 1)\,\lambda}{(2 + \lambda x_1)^2}. \tag{1.18}$$

Here Q_0 is the heat flux in the absence of a field. For strong fields

$$Q = \frac{4\rho c_p g\beta\Theta^2 h^3}{(2 + \lambda x_1)^2\,\nu} \times$$

$$\times \left[\frac{4}{3} + x_1 + \lambda x_1\,(x_1 + 1)\right]\frac{1}{M^2} \quad (M \gg 1), \tag{1.19}$$

(in formulas (1.17)–(1.19) the quantity x_1 is taken to be positive).

For $|x_1| \to 0$ the formulas given in this section change to the corresponding formulas for an isolated channel, found previously [3].

2. **Equilibrium stability.** We now consider the equilibrium stability of a conducting fluid in connected vertical channels in the presence of a transverse magnetic field. If the fluid is heated from below, in the equilibrium state the temperature $T_0 = -Az$, where A is the equilibrium temperature gradient. The normal perturbations are functions of time according to the law $\exp(-\delta t)$, while in the presence of a magnetic field the perturbation decrement δ will, generally speaking, be a complex quantity. In this case the equilibrium of a fluid heated from below may break down

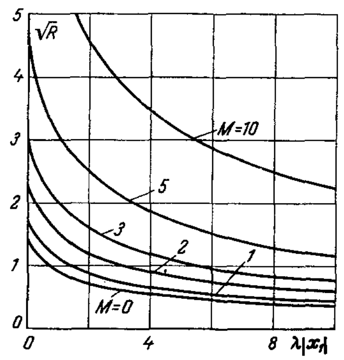

Fig. 3

as the result of the development of both monotonic as well as oscillatory perturbations. However, it is known (see [4,5]) that equilibrium can become critical with regard to oscillatory perturbations only when the inequality $P_m > P$ is satisfied ($P = \nu/\chi$ is the Prandtl number), which as a rule is not so for laboratory conditions. Thus it is of utmost interest to investigate stability with regard to monotonic perturbations for which δ is a real quantity which vanishes at the limit of stability. This case is treated below.

As regards the perturbation structure we assume*:

$$v_x = v_y = 0, \quad v_z = v(x), \quad H_x = H_y = 0,$$

$$H_z = H(x), \quad T = T(x), \quad \nabla p = 0. \qquad (2.1)$$

We write the perturbation equations in dimensionless variables. For the stability problem it is convenient to select h, χ/h, Ah, and $4\pi\sigma\chi H_0 c^{-2}$ as units of distance, velocity, temperature, and field, respectively. The dimensionless equations for steady-state perturbations then assume the form

$$v'' + RT + M^2H' = 0, \quad T'' + v = 0, \quad H'' + v' = 0,$$

$$T_m'' = 0, \quad H_m'' = 0 \qquad (R = g\beta Ah^4/\nu\chi). \qquad (2.2)$$

Here the Rayleigh number R is determined by the equilibrium temperature gradient. We shall solve the perturbation equations for the conditions

$$v = 0, \quad T = T_m, \ \lambda T' = T_m', H = H_m \quad \text{for } x = x_1$$
$$v = 0, \ T' = 0, \quad H = 0 \quad \text{for } x = x_2. \ (2.3)$$

In addition, the flow should occur in a closed circuit. This differs from boundary conditions (1.10), for which the problem of steady-state convection with heating from the side was solved, since it is now assumed that there is no horizontal heat flux in the outer regions of the mass.

As in the case in which there is no field [2], the problem has two types of solution. In solutions of the first type, which in what follows will be called "odd" solutions, the velocity and temperature are odd functions, and the field an even function relative to the coordinate origin. On the other hand in the "even" solutions the velocity and temperature are even functions and the field is an odd function.

We first give the "odd" solutions of the problem (2.2)–(2.3):

$$v = \pm\left[\frac{\cos q(x_2 - x) - \operatorname{ch} p(x_2 - x)}{\cos q(x_2 - x_1) - \operatorname{ch} p(x_2 - x_1)} - \right.$$
$$\left. - \frac{q \sin q(x_2 - x) + p \operatorname{sh} p(x_2 - x)}{q \sin q(x_2 - x_1) + p \operatorname{sh} p(x_2 - x_1)}\right],$$

$$T = \pm\left[\frac{q^{-2}\cos q(x_2 - x) + p^{-2}\operatorname{ch} p(x_2 - x)}{\cos q(x_2 - x_1) - \operatorname{ch} p(x_2 - x_1)} - \right.$$
$$\left. - \frac{q^{-1}\sin q(x_2 - x) - p^{-1}\operatorname{sh} p(x_2 - x)}{q \sin q(x_2 - x_1) + p \operatorname{sh} p(x_2 - x_1)}\right],$$

$$H = \pm\left[\frac{q^{-1}\sin q(x_2 - x) - p^{-1}\operatorname{sh} p(x_2 - x)}{\cos q(x_2 - x_1) - \operatorname{ch} p(x_2 - x_1)} + \right.$$
$$\left. + \frac{\cos q(x_2 - x) - \operatorname{ch} p(x_2 - x)}{q \sin q(x_2 - x_1) + p \operatorname{sh} p(x_2 - x_1)}\right],$$

$$H_m = 2\,\frac{1 + (p^2 - q^2)(2pq)^{-1}\operatorname{sh} 2p \sin 2q - \operatorname{ch} 2p \cos 2q}{(\cos 2q - \operatorname{ch} 2p)(q \sin 2q + p \operatorname{sh} 2p)},$$

$$T_m = H_m \lambda x, \ p = \left\{\left[R + \left(\frac{M^2}{2}\right)^2\right]^{1/2} + \frac{M^2}{2}\right\}^{1/2},$$

$$q = \left\{\left[R + \left(\frac{M^2}{2}\right)^2\right]^{1/2} - \frac{M^2}{2}\right\}^{1/2} \qquad (2.4)$$

(the "plus" and "minus" signs refer to the right- and left-hand channels respectively). The critical values of the Rayleigh numbers which determine the limits of equilibrium stability relative to the odd perturbations are found from the characteristic relation

$$\frac{p^2 + q^2}{2p^2q^2} \times$$
$$\times \frac{p \operatorname{th} 2p + q \operatorname{tg} 2q}{\operatorname{sech} 2p \sec 2q - 1 + (p^2 - q^2)(2pq)^{-1} \operatorname{th} 2p \operatorname{tg} 2q} = \lambda\,|x_1|. \quad (2.5)$$

In the case of the "even" solutions there is no field in the electrically nonconducting interlayer, and the temperature is constant:

$$H_m = 0,$$

$$T_m = \frac{p^2 + q^2}{p^2q^2}\,\frac{q \operatorname{ch} 2p \sin 2q + p \operatorname{sh} 2p \cos 2q}{(\cos 2q - \operatorname{ch} 2p)(q \sin 2q + p \operatorname{sh} 2p)}. \qquad (2.6)$$

The velocity, temperature, and field in the fluid are described by formulas (2.4) with the "plus" sign common to both channels. The spectrum of critical Rayleigh numbers for the "even" solutions is determined by the relations

$$p \operatorname{th} p + q \operatorname{tg} q = 0, \qquad q \operatorname{th} p - p \operatorname{tg} q = 0. \qquad (2.7)$$

The two relations (2.7) correspond to perturbations in which the velocity and temperature are odd and even functions, respectively, relative to the centre of each channel.

Relations (2.5) and (2.7) determine the spectrum of critical Rayleigh numbers. In the "odd" case the critical R numbers depend on two parameters: the Hartmann number M and a parameter $\lambda|x_1|$ which characterizes the thermal coupling of the channels. In the case of "even" solutions the critical R numbers depend on the Hartmann number only; there is no dependence on $\lambda|x_1|$ since in this case there is no thermal interaction between the convective streams in the channels, and "autonomous" circulation takes place in each channel. The spectrum of critical R numbers in the absence of a field has been found and discussed previously [2].

The lower critical level of instability is of greatest interest.

It turns out that this corresponds to the first of the odd solutions. The smallest critical Rayleigh number is given in Fig. 3 as a function of the coupling parameter $\lambda|x_1|$ for several values of the Hartmann number M. For an increase in $\lambda|x_1|$ (as the thermal coupling between the channels decreases) the critical Rayleigh number decreases and tends to zero as $\lambda|x_1| \to \infty$. The magnetic field exerts a stabilizing influence as usual: for a fixed value of $\lambda|x_1|$ the critical Rayleigh numbers increase as the field increases.

The critical Rayleigh numbers corresponding to "even" solutions also increase as the field increases. Thus for solutions which are anti-

*Perturbations which are periodic along the y-axis are not treated here. It is well known that in the case of a layer of infinite extent in the y direction the critical Rayleigh number tends to zero as the wavelength of the perturbations increases [6,7].

symmetric relative to the channel center (the first of relations (2.7)) we have

$$R = R_0 + r\,\mathrm{th}\,r\,(r\,\mathrm{th}\,r - 2)\,M^2 + \ldots \quad (M \ll 1). \tag{2.8}$$

Here $r = R_0^{1/4}$ are the roots of the transcendental equation $\mathrm{tg}\,r + \mathrm{th}\,r = 0$ ($r = 2.365,\ 5.498,\ 8.639\ldots$). For the symmetric solutions (the second of relations (2.7))

$$R = R_0 + \frac{r\,(r - \mathrm{th}\,r)}{\mathrm{th}^2\,r} M^2 + \ldots\ (M \ll 1). \tag{2.9}$$

Here $r = R_0^{1/4}$ is determined from the equation $\mathrm{tg}\,r - \mathrm{th}\,r = 0$ ($r = 3.927,\ 7.069,\ 10.21\ldots$). In the case of strong fields ($M \gg 1$) we have from (2.7)

$$R = {}^1/_4 n^2 \pi^2 M^2 \ (n = 1,\ 2,\ 3\ldots).$$

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